The Runge-Kutta Discontinuous Galerkin Method Applied to the Two-Fluid Plasma System in 2D: Numerics and Preliminary Results

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Overview

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Motivation

1. We are interested in numerical methods for advanced plasma models to better capture physics.
2. Full particle codes are impractical for high-density plasmas.
3. MHD codes ignore potentially important physics and Hall MHD codes require large dissipation for stability.
4. Reduced systems can be compared to the two-fluid model to determine important physics.
Two-Fluid Plasma Model: Fluid Equations

- species continuity
  \[ \partial_t \rho_\alpha + \nabla \cdot \rho_\alpha \mathbf{U}_\alpha = 0 \]

- species momentum
  \[ \rho_\alpha \frac{d \mathbf{U}_\alpha}{dt} + \nabla p_\alpha = \pm \left( \frac{x_0}{r_{gi}} \right) n_\alpha (E + \mathbf{U}_\alpha \times \mathbf{B}) \]

- species energy
  \[ \partial_t e_\alpha + \nabla \cdot \mathbf{U}_\alpha (e_\alpha + p_\alpha) = \pm \left( \frac{x_0}{r_{gi}} \right) n_\alpha \mathbf{U}_\alpha E \]
Two-Fluid Plasma Model: Maxwell’s Equations

- Ampere’s Law

\[ \partial_t E - \left( \frac{c}{v_{thi}} \right)^2 \nabla \times B = - \left( \frac{x_0}{\lambda_d} \right) \left( \frac{r_{gi}}{\lambda_d} \right) (n_i U_i - n_e U_e) \]

- Faraday’s Law

\[ \partial_t B + \nabla \times E = 0 \]

- Poisson’s Equations

\[ \nabla \cdot E = \left( \frac{x_0}{\lambda_d} \right) \left( \frac{r_{gi}}{\lambda_d} \right) (n_i - n_e) \]

- Magnetic Flux

\[ \nabla \cdot B = 0 \]
Numerical Issues

1. The plasma frequency is small and must be resolved.
2. $Dx/c$ is small and must be resolved.
3. $\text{Div } B$ must be satisfied
4. $\text{Div } E = \rho$ must be satisfied
5. The source terms must correctly balance the hyperbolic fluxes near or at equilibrium.
6. Grid resolution must be manageable.
Example of Numerical Error: Div B Errors

With full limiters the second order RKDG scheme is unusable in 2D. Large Div B errors

Without limiters the solution behaves much better.
Example of Numerical Errors: Div E Errors

number density with debye length=1/1000

\[ n_e \quad n_i \]
Other Issues

1. We must differentiate between numerics, solutions that the two-fluid equations predict and solutions that are physical.
2. Running a two-fluid code in a regime dominated by kinetic effects may produce the correct two-fluid solution, but still may not be a physical solution.

Where do we stand now?

Our focus now is on differentiating numerics from the solution predicted by the two-fluid equations.
The Discontinuous Galerkin Method

- The Discontinuous Galerkin Method is a high order extension of finite volume methods.
- High order accuracy is obtained by increasing the accuracy of the polynomial representation of each conserved variable in each cell.
- The stencil does not widen when the accuracy of the scheme is increased.
- Source terms can be treated simply
- High order time accuracy is easy to achieve.
- The method is easy to implement on arbitrary geometries.
The Discontinuous Galerkin Method

\[ U = U_0 + U_x x \]

\[ I - 1 \quad I \quad I + 1 \]
The Discontinuous Galerkin Method

- A general balance law like Maxwell’s equations and the fluid equations is

\[ \frac{\partial q}{\partial t} + \nabla \cdot F = \psi. \]

- Multiply by basis functions and integrate to get,

\[ \int \frac{\partial q}{\partial t} v_h \, dV + \int \nabla \cdot F v_h \, dV = \int_k \psi v_h \, dV. \]

- Integrate by parts,

\[ \int_k \frac{\partial q}{\partial t} v_h \, dV + \int_{\partial K} F \cdot n v_h \, d\Gamma - \int_k F \cdot \nabla v_h \, dV = \int_k \psi v_h \, dV. \]

- Discretize and use Gaussian integration to get

\[ \frac{\partial q_h}{\partial t} V + \sum_e \sum_l w_l F \cdot n v_h \Gamma_e - \sum_m w_m F \cdot \nabla v_h \, V = \sum_m w_m \psi v_h \, V. \]
The Discontinuous Galerkin Method

- For second order space in 2D we use these basis functions.

\[ \{v_h\} = \{v_0, v_x, v_y\} = \{1, \frac{x - x_{ij}}{\Delta x}, \frac{y - y_{ij}}{\Delta y}\} \]

- Update equation for the 0 coefficient.

\[ \frac{\partial q_0}{\partial t} V + \sum_e \sum_l w_l F \cdot n v_0 \Gamma_e = \sum_m w_m \psi v_0 V \]

- Update equation for slope in the x direction.

\[ \frac{\partial q_x}{\partial t} V + 3 \sum_e \sum_l w_l F \cdot n v_x \Gamma_e - 3 \sum_m w_m F \cdot \nabla v_x V = 3 \sum_m w_m \psi v_x V \]

- Update equation for the slope in the y direction.

\[ \frac{\partial q_y}{\partial t} V + 3 \sum_e \sum_l w_l F \cdot n v_y \Gamma_e - 3 \sum_m w_m F \cdot \nabla v_y V = 3 \sum_m w_m \psi v_y V \]
The Discontinuous Galerkin Method – Time Discretization.

1. Time discretization is done using either a 3rd order TVD Runge-Kutta method or a 4th order Non-TVD Runge-Kutta method.
2. Both methods produce similar results, though the 4th order method has a larger stability region.
General Properties of the Discontinuous Galerkin Method Applied to the Two-Fluid System

1. The method is explicit.
2. The method is second order accurate in space.
3. The method is 3rd or 4th order accurate in time.
4. The electron plasma frequency must always be resolved, $Dt < 1/w_{pe}$.
5. The courant condition for electromagnetic waves must be satisfied, $Dt < (1/3)Dx/c$ in 2D using 3rd order RK and $Dt < (1/2)Dx/c$ in 2D using 4th order RK.
Shock Problems

Why do we try and simulate shock waves?

2. Because they excite all wave modes and issues with an algorithm can be identified.
3. Discontinuities are challenging to resolve numerically – if it works on a shock it will work on problems without shocks (hopefully).
4. There are analytic solutions to shock problems (not for the two-fluid system though).
5. They are simple to set up and don’t require complex boundary conditions.
1D Electrostatic Plasma Shock

$\frac{M_i}{M_e} = 1836$, Debye Length \( \sim \frac{1}{1000} \)
1D Electrostatic Plasma Shock with 4000 cells – Resolving the Debye Length
1D Electrostatic Plasma Shock with 4000 Cells – Resolving the Debye Length
1D Electrostatic Plasma Shock with 100 Cells – Without Resolving the Debye Length

![Graph of electron and ion number density at the shock layer](image1)

![Graph of Ex at the shock layer](image2)
1D Electrostatic Plasma Shock with 4000 Cells – Two Temperature Effects
1D Electrostatic Plasma Shock with 4000 Cells – Electron Plasma Waves
1D Electrostatic Plasma Shock – Conclusion

1. Debye length shock structures may develop when the Debye length is resolved.
2. These structures can be explained using the jump conditions for the ion fluid.
3. Even if the Debye length is not resolved, electrostatic shocks propagate at the correct speed and the correct jump conditions are obtained using the MHD equations with B=0.
1D Electromagnetic Plasma Shock

\( \frac{M_i}{M_e} = 1836 \), Debye Length \( \sim 1 \times 10^{-5} \), Ion Larmor Radius \( \sim 1 \times 10^{-3} \)
1D Electromagnetic Plasma Shock – Without Resolving Ion Larmor Radius (1000 Cells)
1D Electromagnetic Plasma Shock – Resolving Ion Larmor Radius (10000 cells)
1D Electromagnetic Plasma Shock - Resolving Ion Larmor Radius (10000 cells)
1D Electromagnetic Plasma Shock – Resolving Ion Larmor Radius (10000 cells)
1D Electromagnetic Plasma Shock-Without Resolving Ion Larmor Radius (1000 cells)
1D Electromagnetic Plasma Shock
-Conclusion

1. At low spatial and temporal resolution (un-resolved ion Larmor radius) the solutions appear MHD-like, though the slow shock travels at the incorrect speed.

2. At high temporal and spatial resolution (resolved ion-Larmor radius) the shock speed is correct though non-MHD oscillations appear and need to be explained.

3. When the ion Larmor radius is resolved, sub-Larmor radius shocks appear and the correct jump conditions for these shocks are obtained using single fluid MHD with B=0
Collisionless Magnetic Reconnection (GEM Challenge): IC’s

Ion Inertial Length = c/wpi ~ 1, Mi/Me = 25

![Graphs showing Electron Jz, Electron and ion n, Total fluid pressure, Bx](image-url)
Collisionless Magnetic Reconnection

|\mid E| on a 128X64 grid at t=15/Wci

|\mid E| on a 200X100 grid at t=15/Wci
Collisionless Magnetic Reconnection

Bx on 128X64 grid with t=15/wci

Bx on 200X100 grid with t=15/wci
Collisionless Magnetic Reconnection

Electron mass density on 128X64 grid at t=15/wci

Electron mass density on 200X100 grid at t=15/wci
Collisionless Magnetic Reconnection

BIRN ET AL.: GEM RECONNECTION CHALLENGE

The diagram shows the reconnected flux over time for different models:

- Full Particle
- Hybrid
- Hall MHD
- MHD
- Two-Fluid

The y-axis represents the reconnected flux, and the x-axis represents time (t).
Collisionless Magnetic Reconnection - Conclusion

1. The two-fluid solution gives similar reconnected magnetic flux to other plasma algorithms.
2. High frequency waves begin to appear outside the reconnection region as resolution increases.
3. Higher resolution runs need to be completed.
2D Cylindrical Electromagnetic Plasma Shock

Ion Larmor Radius $\sim 1 \times 10^{-1}$
Debye Length $\sim 1 \times 10^{-3}$
$\frac{M_i}{M_e} = 1836$
2D Cylindrical Electromagnetic Plasma Shock – 40X40 grid

Peak electron momentum occurs in the shock region (moving counter clockwise about the axis) to support the B field gradient.

The ions contribute little to the azimuthal current.
The region of azimuthal electron current is much thinner at this resolution and eventually breaks up into what looks like turbulence.

All variables are affected including including ion momentum and ion density.
2D Cylindrical Plasma Shock – 400X400 Grid, close up of electron momentum
1. At low resolution (corresponding to high artificial dissipation) the solution appears as we would expect.
2. At high resolution the current layer breaks up. We believe this is a result of a Kelvin Helmholtz instability in the electron fluid, though it may be numerical. More analysis is required.
3. Artificial dissipation may keep a sheared flow stable when it is really unstable.
Conclusion

1. The Discontinuous Galerkin method has been successfully applied to the two-fluid plasma system.
2. It produces results that agree with previous algorithms, but requires a substantially smaller grid.
3. The method produces reasonable results for the electrostatic and MHD shocks, though high frequency oscillations that occur at high resolution need to be investigated.
4. The method produces solutions to the magnetic reconnection problem that agree with the GEM challenge results.
5. Further grid convergence studies need to be performed