Nonlinear plasma dynamics using a Fourier-like expansion:

*Tilts, radial shifts, balloons, and relaxation to plasma Nirvana*

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Familiar objective*

*e.g. of Plasma Dynamics Center

Nirvana — understand a variety of complex plasma dynamics
(1) nonlinear stability (…mode-mode coupling)
(2) relaxation (…fine scale structure);
(3) turbulence cascades (from large to fine scale);
(4) transport

- The tilt mode sucks energy out of the equilibrium state
- Higher-order modes suck energy out of the tilt.
  (ignored in conventional stability theory and computation)

Critical example — FRC tilt stability

The great angst of FRC enthusiasts for 25 years;

Still the make-or-break issue for FRC as a fusion concept.

Question: Does the nonlinear coupling to higher-order modes effectively stabilize the tilt?

...no answer yet
Focus of this seminar: an alternate approach to complex plasma dynamics

Conventional approach: finite difference

Outline of talk:
Alternate approach: Fourier analysis
I. Compare finite-difference and Fourier methods
   Sometimes mixed together (e.g. Fourier analysis in $\theta$ coordinate; FD in $(r,z)$)
II. Fourier-Beltrami approach
III. Early computational results
   A. Tilt stability as baseline
   B. Nonlinear dynamics
Part I

Compare methods:
finite-difference
and
Fourier
finite-difference vs. Fourier

1D heat equation

\[ \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \]

- **Conventional FD method:**
  \[ u(x,t) \rightarrow u_n = u(x_n,t) \]
  \( x_n = \) spatial grid points; \( u_n(t) = u \) at grid points

- **Fourier method:**
  \[ u(x,t) = \sum_{n}^{\infty} u_n(t) \sin(n\pi x) \]
  \( \sin(n\pi x) = \) complete orthogonal basis set
  \( u_n(t) = \) Fourier coefficients

**Summary:** spatial grid (FD) vs modes (Fourier)
Compare FD vs Fourier (cont.)

FD method:
spatial grid

Fourier analysis
(in space)

Spatial resolution same for both: \( \delta = \frac{L}{N} \):

\[ N = \# \text{ grid points (FD)} \text{ or } \# \text{ modes in basis set} \]
Compare FD vs Fourier: SIZE OF COMPUTATIONAL PROBLEM

**SPATIAL** (assume same resolution):
The number of **numerical “entities”** to be computed is **same** for FD and Fourier (within factors of “2” or “π”)  

**TEMPORAL**
Probably the same.  

*The search for the Northwest passage:*
Is there a trick that gets you there faster?  

*So why…?*
Part II

Fourier-Beltrami approach
Fourier-Beltrami analysis

- In a fluid plasma there are two primary vector fields $\mathbf{u}$, $\mathbf{B}$ plus scalar variables ($n$, $p$, …)
- Needed: a vector basis set that is complete and orthogonal
- If the vectors are divergence-free then the Beltrami vectors are available
- If we assume incompressibility ($n = \text{const}$) then both $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \mathbf{B} = 0$

What are Beltrami vectors?
**Definition of Beltrami vectors**

**p.d.e.** \( \nabla \times \mathbf{Y}_k = \Lambda_k \mathbf{Y}_k \)

**B.C.** \( \mathbf{Y} \cdot \mathbf{\hat{n}} = 0 \)

“eigenfunctions of the curl operator”

**orthogonality** \( (1/V) \int d\tau \mathbf{Y}_k \cdot \mathbf{Y}_l = \delta_{kl} \)

**useful property:**

**cross-product** \( \mathbf{Y}_k \times \mathbf{Y}_l = \sum_m M_{klm} \mathbf{Y}_m \)

**expand:** \( \mathbf{B} = \sum_k B_k \mathbf{Y}_k \)

**expansion coefficients**
Beltrami vectors have a Downside

Incompressible. Even so, a common assumption (e.g. NIMROD)

Unique to domain geometry. Must find vector set for each geometry – most serious drawback

STILL... analytic Beltrami vectors available for certain simple geometries:

- *sphere* – e.g. spheromak
- *periodic cylinder* - elongated/compact, e.g. FRC, spheromak
What do Beltrami vectors look like in a periodic cylinder?

Vector components

Bessel functions

\[
Y_r = \left[ \Lambda \frac{mJ_m(\alpha r)}{r} + \kappa \frac{dJ_m(\alpha r)}{dr} \right] \cdot \exp[i(m\theta + \kappa_n z)]
\]

complex phasor

\[
Y_\theta = \left[ \frac{mJ_m(\alpha r)}{r} + \Lambda \frac{dJ_m(\alpha r)}{dr} \right] \cdot i \exp[i(m\theta + \kappa_n z)]
\]

\[
Y_z = \alpha^2 J_m(\alpha r) \cdot (-i) \exp[i(m\theta + \kappa_n z)]
\]

Scale factors and eigenvalue

\[
\kappa_n = \frac{2\pi n}{L} \quad \Lambda = \pm \sqrt{\alpha^2 + \kappa_n^2}
\]

\[
\alpha \rightarrow \alpha \mid \text{determined by B.C.}
\]
Two-fluid dynamics

Motion

\[
\frac{\partial}{\partial t} m_i \mathbf{u} = -\nabla \left( h_i + h_e + m_i u^2 / 2 \right) + T_i \nabla s_i + T_e \nabla s_e + \\
+ \mathbf{u} \times \nabla \times m_i \mathbf{u} + \frac{1}{4\pi n} \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} + \frac{R_i + R_e}{n}
\]

convective inertia

viscous friction

Ohm’s

\[
\frac{\partial}{\partial t} \left( \frac{eA}{c} \right) = \nabla h_e + T_e \nabla s_e + \\
+ \mathbf{u} \times \frac{e\mathbf{B}}{c} - \frac{c^2}{4\pi e^2 n} \left( \nabla \times \frac{e\mathbf{B}}{c} \right) \times \frac{e\mathbf{B}}{c} - \frac{R_e}{n}
\]

resistive friction

Hall effect

u×B

resistive friction

Hall effect

u×B

viscous friction

pressure

convective inertia

j×B force

electric field

electron pressure
Fourier-Beltrami transform

**Motion**
\[
\frac{d}{dt} u_k = -\frac{\Pr \eta c^2}{4\pi} \Lambda_k^2 u_k + \sum_{l,m} M_{klm} \left( \Lambda_m u_l u_m + \Lambda_l \frac{B_l B_m}{4\pi m_l n_l} \right) + \sum_{l,m} u_l - \ell_i^2 \frac{e}{m_i c} \Lambda_l B_l M_{klm} \Lambda_k B_m
\]

viscous friction: \( \Pr = \) Prandtl #

\( \Pr \approx 3 \) for \( \beta \approx 1 \)

convective inertia

j\times B force

Important nonlinear: mode-mode coupling

**Ohm's**
\[
\frac{d}{dt} B_k = -\frac{\eta c^2}{4\pi} \Lambda_k^2 B_k + \sum_{l,m} u_l - \ell_i^2 \frac{e}{m_i c} \Lambda_l B_l M_{klm} \Lambda_k B_m
\]

viscous friction

single-fluid recovered for \( l_i \to 0 \)

resistive friction

Hall effect: \( l_i = \) ion skin depth

\( u \times B \)
Part III

Early computational results

A. Stability
B. Nonlinear dynamics
Stability properties (standard MHD) are well known; serves as a check on FB method

Standard MHD:

- Zero-order: axisymmetric; no flow
- First-order: weak disturbance

Apply to tilt instability
Formalism for tilt-like modes

Zero-order terms:

- nonevolving & axisymmetric
- \( B_k^{(0)} = \text{const} \) for \( m = 0 \) (axisymmetric)
- \( B_k^{(0)} = 0 \) for \( m \neq 0 \)
- all \( u_k^{(0)} = 0 \) (static equilibrium)

First-order terms

- harmonic time dependence \( \partial / \partial t = -i \omega \)
- \( B_k^{(1)}(t), \quad u_k^{(1)}(t) \) for \( m = 1 \) (tilt-like)
- \( B_k^{(1)} = 0, \quad u_k^{(1)} = 0 \) for all \( m \neq 1 \)
Minimal basis set to represent tilt

Requires three basis elements:

#1  $k = (1,0,1)$ simple equilibrium

#2  $k = (1,1,1)$ tilt-like flow disturbance

#3  $k = (1,1,2)$ tilt-like field disturbance
Simple tilt (3-basis set)

growth rate vs elongation

Even with minimal basis set, magnitude and scaling of growth rate not bad.
(III-B) Nonlinear evolution: early results

Diagnostic questions
How well are the constants of motion preserved?
And others…

Physics questions
How quickly do low-order modes cascade to higher order? (e.g. do low-order modes get “turned off”)
Does angular momentum appear to play a role in stability?
What does the “turbulence spectrum” look like when things settle down?
And others…
How well are constants of motion preserved?
(if the dissipation is turned off)

“Six pack” basis set

“72-pack” set (~ 4x4x4 grid)

Why? Probably not the size of the basis set but the selection of modes.

to be determined.
Nonlinear dynamics

Initial condition: static “FRC” and small tilt-like disturbance

magnetic fields (rms)  flows (rms)

Spontaneous flow generation with large overshoot

Is there a steady, slightly non-axisymmetric state?

What is the fluctuation level?
Does angular momentum play a role?

In static equilibria (zero ang. mom.): apparently not
In rigidly rotating equilibria: perhaps not ??

However in MES…
(Geren-Steinhauer, subm to PoP 2004)

non-zero rotation

- increases energy of the MES
- assures that MES is axisymmetric
Fourier-Beltrami expansion is an alternative to finite-difference methods. 

Disadvantage: it is geometry-specific

No computational advantage. HOWEVER...tricks may be available to effectively reduce the basis set size without giving up essential physics (ideas forthcoming)

Interesting nonlinear phenomena already found in early computations

Stay tuned